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Best L_1-Approximation by Generalized Convex Functions

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Let $\{u_0, ..., u_{n-1}\}$ be an ECT-system and let $C(u_0, ..., u_{n-1})$ be its generalized convexity cone. We give a precise description of best L_1-approximations to continuous functions from $C(u_0, ..., u_{n-1})$. These best approximations are, piecewise, certain extremal Chebyshevian splines, which are obtained by applying results from moment theory for Chebyshev systems.

Let $\omega_1, ..., \omega_n$ $(n \geqslant 2)$ be n-times continuously differentiable, strictly positive functions defined on the real interval (α, β) and let $a \in (\alpha, \beta)$ be fixed. We define an *extended complete Chebyshev system* (ECT-system) $\{u_0, ..., u_{n-1}\}$ by

$$u_0(x) = \omega_n(x),$$

$$u_1(x) = \omega_n(x) \int_a^x \omega_{n-1}(\xi_{n-1})\, d\xi_{n-1},$$

$$\vdots$$

$$u_{n-1}(x) = \omega_n(x) \int_a^x \omega_{n-1}(\xi_{n-1}) \int_a^{\xi_{n-1}} \cdots \int_a^{\xi_2} \omega_1(\xi_1)\, d\xi_1 \cdots d\xi_{n-1}.$$

This is, in fact, an ECT$^+$-system; i.e., all of the Wronskians $W[u_0, ..., u_i]$ are strictly positive for $i = 0, ..., n-1$. ECT-systems, which are related to the notion of extended total positivity, were extensively investigated in [5, Chap. XI; 4, Chap. 6]; another good source is [9, Chap. 9]. In approximation theory their importance lies in the fact that they share many of the properties of algebraic polynomials (which may be constructed in this way by making all the ω_i constant). Associated with this system of functions is a sequence of differential operators: $L_i := (1/\omega_{n-i}) D \cdots D(1/\omega_n)$ $(i = 0, ..., n-1)$, $D := d/dx$. Setting $L := D(1/\omega_1) \cdots D(1/\omega_n)$ we see that

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$U := \operatorname{span}\{u_0, ..., u_{n-1}\}$ is the nullspace of the disconjugate differential operator L.

DEFINITION 1. A function φ is generalized convex with respect to the ECT-system $\{u_0, ..., u_{n-1}\}$ if the "augmented generalized Vandermonde" determinant

$$\begin{vmatrix} u_0(x_0) & \cdots & u_0(x_n) \\ u_1(x_0) & \cdots & u_1(x_n) \\ \vdots & & \vdots \\ u_{n-1}(_0) & \cdots & u_{n-1}(x_n) \\ \varphi(x_0) & \cdots & \varphi(x_n) \end{vmatrix}$$

is nonnegative for all $\alpha < x_0 < \cdots < x_n < \beta$.

This set of generalized convex functions is a convex cone and is denoted by $C(u_0, ..., u_{n-1})$. Generalized convex functions enjoy certain differentiability properties as described in [5, 1]. In particular, their $(n-2)$nd derivatives are continuous. If we define

$$L_{n-1}^{\pm} := \frac{1}{\omega_1} D^{\pm} \frac{1}{\omega_2} D \cdots D \frac{1}{\omega_n}$$

then, for $\varphi \in C(u_0, ..., u_{n-1})$, $L_{n-1}^{+}\varphi$ is right-continuous and nondecreasing and $L_{n-1}^{-}\varphi$ is left-continuous and nondecreasing. To a generalized convex function φ we may associate a nonnegative, regular Borel measure μ on (α, β) by setting $\mu([c, d]) = L_{n-1}^{+}\varphi(d) - L_{n-1}^{-}\varphi(c) \geq 0$. Then on any interval $[a, b] \subset (\alpha, \beta)$ we have the representation

$$\varphi(x) = u(x) + \int_{[a,b]} K_n(x, t)\, d\mu(t), \qquad x \in [a, b], \tag{1}$$

with

$$u(x) = \sum_{i=0}^{n-2} (L_i\varphi)(a) u_i(x) + (L_{n-1}^{-}\varphi)(a) u_{n-1}(x),$$

and K_n as defined below. This representation may be extended to all of $[\alpha, \beta)$ only if both of $L_{n-1}^{\pm}\varphi$ are bounded in (α, β). However, the set of generalized convex functions with such representations on $[\alpha, \beta]$ is (uniformly) dense in $C(u_0, ..., u_{n-1})$ [5].

DEFINITION 2. Let $f \in L_1([\alpha, \beta]$ and $g \in C(u_0, ..., u_{n-1}) \cap L_1[\alpha, \beta]$ be given. Then g is a best L_1-approximation for f from $C(u_0, ..., u_{n-1})$ if

$$\|f-g\|_1 := \int_\alpha^\beta |f(x)-g(x)|\, dx = \inf\{\|f-\varphi\|_1 : \varphi \in C(u_0, ..., u_{n-1})\}.$$

The concept of the "dual system" to an ECT-system will be important in our considerations. The dual system is a basis for the nullspace of the formal adjoint of L, which is given by $L^* = (-1)^n(1/\omega_n)D \cdots (1/\omega_1)D$, $D := d/dt$. One such basis is

$$u_0^*(t) \equiv 1,$$

$$u_1^*(t) = \int_t^b \omega_1(\xi_1)\, d\xi_1,$$

$$\vdots$$

$$u_{n-1}^*(t) = \int_t^b \omega_1(\xi_1) \int_{\xi_1}^b \cdots \int_{\xi_{n-2}}^b \omega_{n-1}(\xi_{n-1})\, d\xi_{n-1} \cdots d\xi_1.$$

Now $\{u_0^*, ..., u_{n-1}^*\}$ is an ECT-system, but not an ECT^+-system; however, it may be transformed by a change of basis into the following (dual) ECT^+-system:

$$v_0(t) \equiv 1,$$

$$v_1(t) = \int_a^t \omega_1(\xi_1)\, d\xi_1,$$

$$\vdots$$

$$v_{n-1}(t) = \int_a^t \omega_1(\xi_1) \int_a^{\xi_1} \cdots \int_a^{\xi_{n-2}} \omega_{n-1}(\xi_{n-1})\, d\xi_{n-1} \cdots d\xi_1.$$

Both of these dual systems play a role in the proof of Theorem 1.

In order to introduce the notion of Chebyshevian spline, we first define the *fundamental kernel*, the Green function for L([9], [5]):

$$K_n(x, t) = \omega_n(x) \int_t^x \omega_{n-1}(\xi_{n-1}) \int_t^{\xi_{n-1}} \cdots \int_t^{\xi_2} \omega_1(\xi_1)\, d\xi_1 \cdots d\xi_{n-1}, \qquad t \leqslant x$$

$$= 0, \qquad t > x.$$

A *Chebyshevian spline* is a function of the form

$$s(x) = u(x) + \sum_{i=0}^{k+1} \alpha_i K_n(x, \tau_i), \tag{2}$$

with *knots* $\alpha<\tau_0<\cdots<\tau_{k+1}<\beta$ and $u\in U$. Like polynomial splines, Chebyshevian splines are $(n-2)$-times continuously differentiable and have jump discontinuities in the $(n-1)$st derivative at the τ_i (provided the corresponding α_i is not zero).

THEOREM 1. *Let ω_i, u_i, and (α, β) be as above and let $g\in C(u_0, ..., u_{n-1})$ be given. If $[a, b]$ is contained in (α, β) then there are Chebyshevian splines $\underline{s}$ and $\bar{s}$ on (α, β), such that if $\varphi\in C(u_0, ..., u_{n-1})$ coincides with g in $(\alpha, \beta)\backslash(a, b)$ then*

$$\underline{s}(x)\leqslant\varphi(x)\leqslant\bar{s}(x), \qquad x\in[a, b].$$

These extremal splines have the form (2) *with*

$$u(x)=\sum_{i=0}^{n-2}(L_i g)(a)(u_i(x)+(L_{n-1}^{-} g)(a)u_{n-1}(x).$$

Furthermore, $\underline{s}$ and $\bar{s}$ are unique in $[a, b]$ and we have:

n even: *For $\bar{s}$, $k=n/2-1$, $a=\tau_0<\tau_1<\cdots<\tau_{k+1}=b$; for $\underline{s}$, $k=n/2$, $a<\tau_1<\cdots<\tau_k<b$, $\alpha_0=\alpha_{k+1}=0$;*

n odd: *For $\bar{s}$, $k=(n-1)/2$, $\alpha=\tau_0<\tau_1<\cdots<\tau_k<b$, $\alpha_{k+1}=0$; for $\underline{s}$, $k=(n-1)/2$, $a<\tau_1<\cdots<\tau_{k+1}=b$, $\alpha_0=0$.*

The Chebyshevian splines $\underline{s}$ and $\bar{s}$ will be referred to as *lower* and *upper extremal splines*, respectively. Thus, $\underline{s}$ and $\bar{s}$ form the boundary of the "interpolating envelope" on $[a, b]$ of generalized convex functions that agree with g outside of (a, b).

EXAMPLES. For $n=2$ and $\omega_i\equiv 1$, a generalized convex function g is convex in the usual sense. In this case the upper extremal spline $\bar{s}$ for an interval $[a, b]$ is just the linear polynomial interpolant, and the lower extremal spline $\underline{s}$ is the piecewise linear function with at most one knot, which agrees with g at the endpoints and satisfies $\underline{s}'(a)=g'_-(a)$, $\underline{s}'(b)=g'_+(b)$. The interpolating envelope in this case is, thus, a triangle (see [3]).

In the next example, $n=3$, $\omega_i\equiv 1$, and $[a, b]=[-1, 1]$, so that $U=\text{span}\{1, (x+1), (x+1)^2/2\}$ and $g(x)=x^3-x$ is generalized convex ("3-convex"). The upper and lower extremal splines in this case are quadratic polynomial splines with two knots each. Figure 1 shows $\underline{s}$ and $\bar{s}$ restricted to $[-1, 1]$.

Proof of Theorem 1. If φ agrees with g outside of (a, b) then φ may be

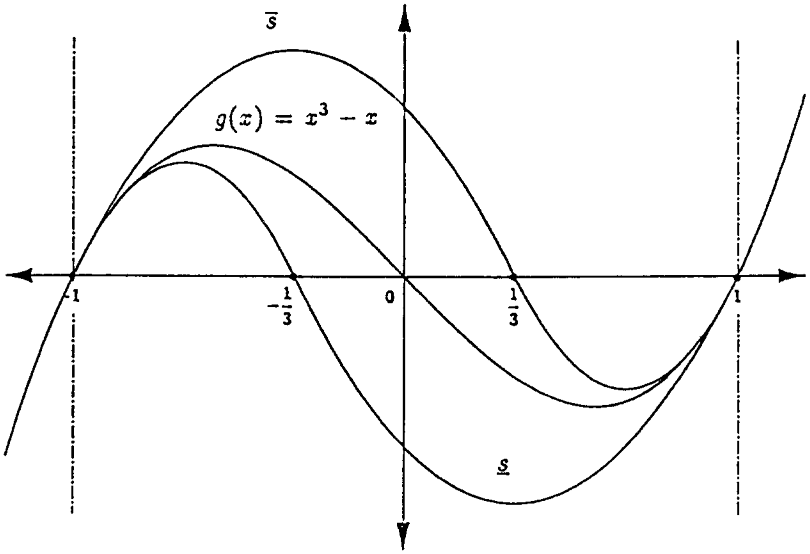

FIG. 1. Example of an interpolating envelope for $n=3$.

represented as in (1) with u as in the statement of the theorem, and with μ satisfying

$$\int_{[a,b]} u_0^*(t)\,d\mu(t)=\int_{[a,b]} 1\,d\mu(t)=\mu([a,b])=L_{n-1}^+ g(b)-L_{n-1}^- g(a). \qquad (3)$$

Moreover,

$$\int_{[a,b]} u_{n-1-i}^*(t)\,d\mu(t)=\int_{[a,b]} (L_iK_n)(b,t)\,d\mu(t)$$
$$=L_i(\varphi-u)(b)=L_i(g-u)(b) \qquad (i=0,\ldots,n-2).$$

Thus, this interpolation problem may be transformed into a *moment problem* for the dual ECT$^+$-system $\{v_0,\ldots,v_{n-1}\}$:

$$\int_a^b v_i(t)\,d\mu(t)=c_i \qquad (i=0,\ldots,n-1). \qquad (4)$$

Let M_c denote the set of nonnegative Borel measures for which the moment conditions (4) are satisfied. Since an ECT-system is also a Chebyshev system, by the Markov–Krein Theorem [5, 6, 7] there are unique extremal measures $\underline{\mu}$ and $\bar{\mu}$ in M_c such that for all $\mu \in M_c$ and all $\psi \in C(v_0, ..., v_{n-1})$

$$\int_{[a,b]} \psi(t)\, d\underline{\mu}(t) \leqslant \int_{[a,b]} \psi(t)\, d\mu(t) \leqslant \int_{[a,b]} \psi(t)\, d\bar{\mu}(t). \tag{5}$$

These extremal measures are discrete measures with mass distributed as follows:

n even: $\bar{\mu}$ has mass at a and at b, and at $k = n/2 - 1$ intermediate points; $\underline{\mu}$ has mass only at $k = n/2$ intermediate points;

n odd: $\bar{\mu}$ has no mass at a, and has mass at b and at $k = (n-1)/2$ intermediate points; $\underline{\mu}$ has no mass at b and has mass at a and at $k = (n-1)/2$ intermediate points.

It should be mentioned that these conclusions are valid provided μ has sufficient number of mass points ("index n" [6]); otherwise, g is a spline on $[a, b]$ and $\underline{s}$, $\bar{s}$ and g all coincide.

Functions $\psi \in C(v_0, ..., v_{n-1})$ have the representation

$$\psi(t) = v(t) + \int_{[a,b]} (-1)^n K_n(x, t)\, d\mu(x), \qquad t \in [a, b],$$

with $v \in \operatorname{span}\{v_0, ..., v_{n-1}\}$ and μ a nonnegative Borel measure. In particular, for fixed x the function $\psi(t) := (-1)^n K_n(x, t)$ is an element of $C(v_0, ..., v_{n-1})$ and therefore (5) holds. The proof is now completed by adding $(-1)^n u(x)$ to each term of (5) and setting

$$\underline{s}(x) := u(x) + \int_{[a,b]} K_n(x, t)\, d\underline{\mu}(t), \qquad \bar{s}(x) := u(x) + \int_{[a,b]} K_n(x, t)\, d\bar{\mu}(t)$$

if n is even, and vice versa if n is odd. ∎

Remark 1. It can be shown that the knots of $\underline{s}$ and $\bar{s}$ strictly interlace.

A simple consequence of (1) and the definition of K_n is the following estimate:

PROPOSITION 1. *Let the conditions and conclusions of Theorem* 1 *prevail. Then, for all $a \leqslant r \leqslant q \leqslant b$ and $x \in [r, q]$,*

$$0 \leqslant \bar{s}(x) - \underline{s}(x) \leqslant K \frac{(q-r)^{n-1}}{(n-1)!} \prod_{i=1}^{n} m_i,$$

where $K = L_{n-1}^{+} g(b) - L_{n-1}^{-} g(a) < \infty$ and $m_i = \max_{[a,b]} |\omega_i|$.

We now prove the main theorem of this paper.

THEOREM 2. *Let f be continuous in (α, β) and let g be generalized convex on (α, β) with respect to $\{u_0, ..., u_{n-1}\}$. If g is a best L_1-approximation to f then in every closed subinterval I of (α, β) there exist disoint open sets E_1, E_2 such that $g<f$ and g is an upper extremal Chebyshevian spline on each connected component of E_1, $g>f$ and g is a lower extremal Chebyshevian spline on each connected component of E_2, and $g=f$ in $I\backslash(E_1 \cup E_2)$.*

Proof. Let (a, b) be a connected component of the (relatively) open set $I \cap \{f>g\}$ and let $[c, d] \subset (a, b)$ be arbitrary. Let $u[c, d]$ denote the generalized convex function that coincides with the upper extremal spline $\bar{s}$ for g on $[c, d]$ and equals g outside of $[c, d]$. For $\lambda > 0$ set

$$g_\lambda := (1-\lambda)\, g + \lambda u[c, d].$$

Thus, $g_\lambda \equiv g$ outside of $[c, d]$. We have

$$0 \leqslant g_\lambda - g = \lambda(u[c, d] - g,$$

hence for small enough λ we have $f > g_\lambda \geqslant g$ in $[c, d]$. If there exists a point $y \in [c, d]$ such that $u[c, d](y) > g(y)$, then $\|f-g_\lambda\|_1 < \|f-g\|_1$, a contradiction to the assumption that g is a best approximation. The proof is completed by applying an analogous argument to connected components of $I \cap \{f<g\}$. ∎

Remark 2. Suppose that the ω_i are bounded. If f is bounded then its best approximation g must be bounded as well. Otherwise, by extrapolation of g to an endpoint by an element of U we can construct a closer approximation to f.

We close with a few comments. For the case when all ω_i are constant (polynomial case), a description of the upper and lower extremal splines was given by Popoviciu [8]. The extremal splines that form the interpolating envelope for a sequence of *distinct* points in the general case were described by Burchard [2, 1]. In this case moment theory for weak Chebyshev systems is needed (see also [7]). Thus, our technique in proving Theorem 1 cannot be considered new. Our proof is, however, made somewhat shorter than previous ones by using the precise results from moment theory expounded in [6]. Outside of the preliminary findings in [3, 10], we are unaware of results similar to Theorem 2 in the literature.

We have not considered the problems of existence and uniqueness in this paper. Based on our experience with convex and n-convex functions, we conjecture that every function, continuous on a compact interval, has a unique, continuous best generalized convex L_1-approximation.

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